ALGEBRAIC NUMBER THEORY FINAL EXAM

This exam is of **60 marks** and is **4 hours long** - from 10 am to 2pm. Please **read all the questions carefully**. Please feel free to use whatever theorems you have learned in class after stating them clearly. You may also refer to the books by

- K. Ireland and M. Rosen - **A Classical Introduction to Modern Number Theory**.
- I.N. Stewart and D.O. Tall - **Algebraic Number Theory**

If you have any questions please call me at **+91 98804 59642** or email me at **rameshsreekantan@gmail.com**. Please sign the following statement and scan this sheet along with the rest.

I have not used any unfair or illegal means to answer any of the questions in this exam.

Name: **Signature:**

1. Consider the biquadratic field $K = \mathbb{Q}(\sqrt{D}, \sqrt{D'})$, where $D, D' \in \mathbb{Z}$ not squares.

a. How many quadratic subfields does it have? What are they? 5
b. What are the possible ranks of the group of units $\mathfrak{O}_K^*$ of the ring of integers $\mathfrak{O}_K$? 5

2. Let K be the field $\mathbb{Q}(\sqrt{3}, \sqrt{-1})$.

a. Find a primitive element for K. 2
b. Find an integral basis for $\mathfrak{O}_K$ and compute Δ_K. 8
c. What is the class number of $\mathfrak{O}_K$? Justify your answer. 5
d. What is the rank of $\mathfrak{O}_K^*$, the group of units? 2
e. What are the torsion elements in $\mathfrak{O}_K^*$? 3
f. Find a unit of infinite order, if it exists. 5

3. Consider the cyclotomic field $K = \mathbb{Q}(\zeta_5)$ where $\zeta_5 = e^{\frac{2\pi i}{5}}$. Let $\mathfrak{O}_K$ be its ring of integers.

a. Factorise 5 into prime ideals. 5
b. Factorise 7 into prime ideals. 5
c. Factorise 11 into prime ideals. 5
d. What is the class number of $\mathfrak{O}_K$? Justify your answer. 5
e. What is the quadratic subfield of K? 5